华东师范大学期中试卷 A 卷
2024-2025 学年第二学期
参考答案及评分标准

课程名称：软件工程数学

学生姓名：＿＿＿＿＿＿＿＿　学　　号：＿＿＿＿＿＿＿＿

专　　业：软件工程　年级/班级：2024 级

课程性质：公共必修、公共选修、<u>专业必修</u>、专业选修

一	二	三	四	五	六	七	八	九	十	总分	阅卷人签名

一、Logic (5 questions, 35 points)

1. (6 points) Determine the truth value of each of these statements if the universe of discourse for all variables consists of all integers.

    1. $\forall x \exists y (x + y = 1)$
    2. $\exists x \exists y (x + 2y = 2 \ \wedge \ 2x + 2y = 5)$
    3. $\exists x \forall y (x < y^2)$

1. (1) T　(2) F　(3) T

2. (5 points) 证明下面的公式（不用真值表方法）

$$\neg(p \leftrightarrow q) \equiv ((p \vee q) \wedge \neg(p \wedge q))$$

2. 证明:

左边 $\equiv \neg((p\to q)\wedge(q\to p))$ 等价等值式

$\equiv \neg((\neg p\vee q)\wedge(\neg q\vee p))$ 蕴含等值式

$\equiv \neg(\neg p\vee q)\vee\neg(\neg q\vee p)$ 德摩根律

$\equiv (p\wedge\neg q)\vee(q\wedge\neg p)$ 德摩根律

$\equiv (p\vee q)\wedge(p\vee\neg p)\wedge(\neg q\vee q)\wedge(\neg q\vee\neg p)$ 分配律

$\equiv (p\vee q)\wedge\neg(p\wedge q)$

$\equiv$ 右边

3. (9 points) 将下列命题符号化（要求使用全总个体域）

1. 火车比轮船快
2. 在北京工作的人未必都是北京人
3. 部分序集 $(A,\leq)$ 没有最大元

3. (1) 令 $F(x)$: x是火车

$G(y)$: y是轮船

$H(x,y)$: x比y快

则 $\forall x\forall y(F(x)\wedge G(y)\to H(x,y))$

(2) 令 $P(x)$: x是北京人

$Q(x)$: x在北京工作

则 $\neg\forall x(Q(x)\to P(x))$

(3) 令 $P(x)$: $x\in A$

$Q(x,y)$: $x\leq y$

则 $\neg\exists x(P(x)\wedge\forall y(P(y)\to Q(y,x)))$

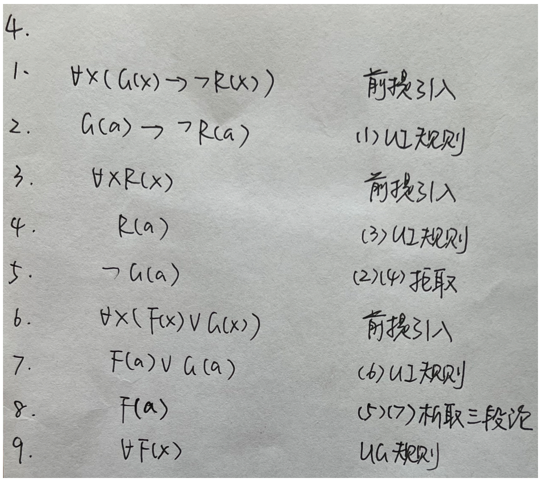
4. (7 points) 试证明下面的推理为有效推理

前提：$\forall x(F(x) \vee G(x))$

$\forall x(G(x) \rightarrow \neg R(x))$

$\forall x R(x)$

结论：$\forall x F(x)$

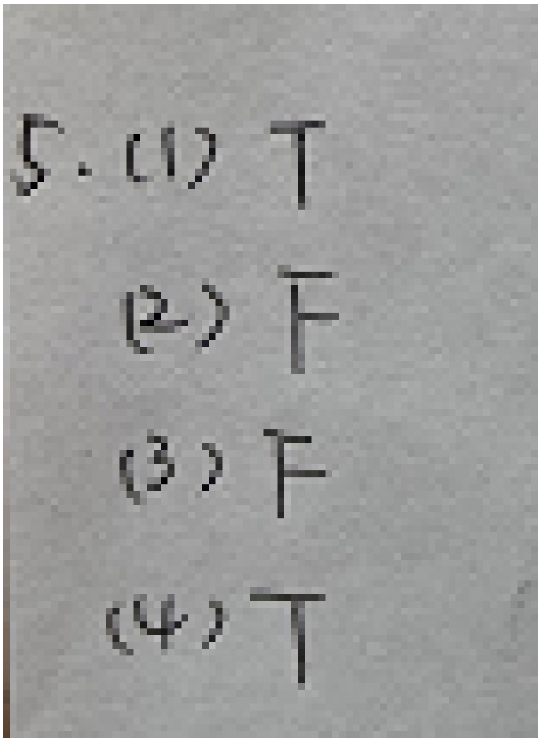


4.

1.	$\forall x(G(x) \rightarrow \neg R(x))$	前提引入
2.	$G(a) \rightarrow \neg R(a)$	(1) UI规则
3.	$\forall x R(x)$	前提引入
4.	$R(a)$	(3) UI规则
5.	$\neg G(a)$	(2)(4) 拒取
6.	$\forall x(F(x) \vee G(x))$	前提引入
7.	$F(a) \vee G(a)$	(6) UI规则
8.	$F(a)$	(5)(7) 析取三段论
9.	$\forall F(x)$	UG规则

5. (8 points) 判断如下的结论是否正确

1. $\forall x(A(x) \wedge B(x)) \equiv \forall x A(x) \wedge \forall x B(x)$
2. $\exists x(A(x) \wedge B(x)) \equiv \exists x A(x) \wedge \exists x B(x)$
3. $\forall x(A(x) \vee B(x)) \equiv \forall x A(x) \vee \forall x B(x)$
4. $\exists x(A(x) \vee B(x)) \equiv \exists x A(x) \vee \exists x B(x)$


5. (1) T
(2) F
(3) F
(4) T

二、Sets and Functions (3 questions, 20 points)

6. (6 points) Let A, B and C be sets. Prove that

$$(A - B) \cup C = A - (B - C) \quad \text{if and only if} \quad C \subseteq A$$

Solution.

$\Longrightarrow$: if $(A - B) \cup C = A - (B - C)$, then we have

$$C \subseteq (A - B) \cup C = A - (B - C) \subseteq A.$$

$\Leftarrow$: if $C \subseteq A$, then we have

$$\begin{aligned}(A - B) \cup C &= (A \cap \overline{B}) \cup C \\ &= (A \cup C) \cap (\overline{B} \cup C) \\ &= A \cap (\overline{B} \cup C) \qquad // \text{ because } C \subseteq A \\ &= A \cap \overline{B \cap \overline{C}} \\ &= A \cap \overline{B - C} \\ &= A - (B - C).\end{aligned}$$

7. (8 points) Determine whether these statements are true or false.

   1. $\emptyset \in \{\{\emptyset\}\}$.
   2. If A and B are both uncountable sets, then $A \cap B$ is also uncountable.
   3. The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$ is a function which is neither onto nor one-to-one.
   4. The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 2 - 2x$ is a function which is both onto and one-to-one.

Solution.

1. False.
2. False. If $A = \mathbb{R}^+$ and $B = \mathbb{R}^-$, then $A \cap B = \emptyset$.
3. False. f is not a function since it is undefined at $x = 0$.
4. True.

8. (6 points) Suppose A is an infinite set. Prove that A has a proper subset B (i.e., $B \subset A$)

such that $|A| = |B|$.

Solution. Since A is infinite, it has a infinite countable subset $A_0 \subseteq A$. Suppose $A_0 = \{a_1, a_2, a_3, \dots\}$. Then we let $B = A - \{a_1\}$. We can construct a bijection f from A to B as:

$$f(x) = \begin{cases} x & x \in A - A_0 \\ a_{i+1} & x = a_i \end{cases}.$$

Thus we have $|A| = |B|$ and complete the proof.

三、Relations (4 questions, 45 points)

9. (10 points) 设 $R \subseteq A \times A$，分别定义以下操作：

- $r(R)$：对 R 取自反闭包
- $s(R)$：对 R 取对称闭包
- $t(R)$：对 R 取传递闭包

（1）请考虑两个组合操作：

$$R_1 = t(s(r(R)))$$

$$R_2 = s(t(r(R)))$$

请证明或举反例说明：是否恒有 $R_1 = R_2$？若成立请证明；若不成立，请给出一个集合 A 和关系 R 的具体例子，并分别写出 R_1 与 R_2。

（2）是否存在一个唯一的最小关系 $S(R \subseteq S)$，同时满足以下三个性质：自反性、对称性、传递性。如果存在，请说明该关系 S 的构造方法；如果不存在，请说明原因。

(1) 不成立。举例 $A=\{1,2,3\}$，$R=\{(1,3),(2,3)\}$。$R_1=t(s(r(R)))=\{(1,1),(2,2),(3,3),(1,3),(2,3),(3,1),(3,2),(1,2),(2,1)\}$；$R_2=s(t(r(R)))=\{(1,1),(2,2),(3,3),(1,3),(2,3),(3,1),(3,2)\}$，$R_2$ 不包含 (1,2) 和 (2,1)，两者不相等。

（2）是否存在一个唯一的最小关系 S ($S\subseteq R$)，同时满足以下三个性质：自反性、对称性、传递性。如果存在，请说明该关系 S 的构造方法；如果不存在，请说明原因。

Solution:

存在唯一的最小关系 S，即 R 的等价闭包。构造方法为：先取 R 的自反闭包 r(R)，再取对称闭包 s(r(R))，最后取传递闭包 t(s(r(R)))。

10. (10 points) Considering relations on the set $A=\{1,2,3,\ldots,100\}$ consisting of the first 100 positive integers, please answer the following questions:

(a) How many nonzero entries does the matrix representing the relation have if $R=\{(a,b)\mid a>b\}$;

(b) How many relations are there on set A that are asymmetric （非对称）

(c) How many relations are there on set A that are both symmetric and antisymmetric （对称且反对称）

(1) 4950

(2) $3^{\frac{100^2-100}{2}}=3^{4950}$

(3) 2^{100} （仅包含自环，每个元素可选是否包含自环。）

11. (12 points) Given a function $f:X\to Y$, define the relation

$$R=\{(x_1,x_2)\mid f(x_1)=f(x_2),x_1,x_2\in X\}$$

(a) Prove that the relation R is an equivalence relation.

(b) Give the corresponding set B of all equivalence classes.

(c) Define the function $g : B \to f(X)$ by $g([x]) = f(x)$. Prove that g is a bijective function.

证明 (a) Prove that the relation R is an equivalence relation.

To prove that R is an equivalence relation, we need to show that it satisfies the three properties of reflexivity, symmetry, and transitivity.

(1) Reflexivity: For any $x \in X$, we have $f(x) = f(x)$. Therefore, $(x, x) \in R$. This shows that R is reflexive.

(2) Symmetry: If $(x, y) \in R$, then $f(x) = f(y)$. This implies $f(y) = f(x)$, so $(y, x) \in R$. This shows that R is symmetric.

(3) Transitivity: If $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$. This implies $f(x) = f(z)$, so $(x, z) \in R$. This shows that R is transitive.

Since R satisfies all three properties, it is an equivalence relation.

(b) Give the corresponding set B of all equivalence classes.

The set of equivalence classes B is given by the partition of X induced by the relation R. Each equivalence class consists of all elements in X that are related to each other under R. Formally, the set of equivalence classes B is:

$$B = \{[x] \mid x \in X\}$$

where $[x]$ denotes the equivalence class of x, defined as:

$$[x] = \{y \in X \mid (x, y) \in R\} = \{y \in X \mid f(x) = f(y)\}$$

(c) Prove that $g : B \to f(X)$ is a bijective function.

Define the function $g : B \to f(X)$ by:

$$g([x]) = f(x)$$

We need to show that g is both injective (one-to-one) and surjective (onto).

(1). Injectivity: Suppose $g([x]) = g([y])$. This means $f(x) = f(y)$. By the definition of the equivalence classes, x and y are in the same equivalence class, i.e., $[x] = [y]$. Therefore, g is injective.

(2). Surjectivity: For any $y \in f(X)$, there exists some $x \in X$ such that $f(x) = y$. Then, $g([x]) = f(x) = y$. This shows that every element in $f(X)$ is the image of some equivalence class in B. Therefore, g is surjective.

Since g is both injective and surjective, it is a bijective function.

12. (13 分) 设 Π_n 表示集合 $S_n = \{1, 2, \ldots, n\}$ 的所有划分构成的集合。给定集合 S_n 上的两个划分 $P_1 = \{A_1, A_2, \ldots, A_r\}$ 和 $P_2 = \{B_1, B_2, \ldots, B_s\}$, 若对于每个 A_j 均有某个 B_k, 使 $A_j \subseteq B_k$, 则称 P_1 是 P_2 的加细, 记为 $P_1 \preceq P_2$.

(a) 证明: 集合 $(\Pi_n, \preceq)$ 是一个偏序集。

(b) 对于 $n = 3$，画出偏序集 $(\Pi_n, \preceq)$ 的哈斯图。

(c) 对于 $n = 5$，求 $\{P_1, P_2\}$ 的最大下界和最小上界，其中 $P_1 = \{\{1, 2\}, \{3\}, \{4, 5\}\}$，$P_2 = \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$。

(a) To show that $(\Pi_n, \preceq)$ is a poset, we need to verify three properties:

1. **Reflexivity**: For any partition $P \in \Pi_n$, clearly every block of P is a subset of itself, so $P \preceq P$.
2. **Antisymmetry**: Let $X \in P_1$. Since $P_1 \preceq P_2$, there exists $Y \in P_2$ such that

$X \subseteq Y$. Since $P_2 \preceq P_1$, there exists $Z \in P_1$ such that $Y \subseteq Z$. But $X \subseteq Y \subseteq Z$ and $X, Z \in P_1$. Since the sets in a partition are non-overlapping, $X = Y = Z$. Similarly, for any set in P_2, it is equal to a set in P_1. Thus, $P_1 = P_2$, and the relation is antisymmetric.

3. **Transitivity**: If $P_1 \preceq P_2$ and $P_2 \preceq P_3$, then any block of P_1 is contained in some block of P_2, which in turn is contained in some block of P_3. Thus $P_1 \preceq P_3$.

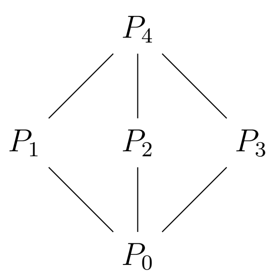
Therefore, $(\Pi_n, \preceq)$ is a poset.

(b) Hasse diagram for Π_3:

For $n = 3$, the set $S_3 = \{1, 2, 3\}$ has 5 partitions:

- $P_0 = \{\{1\}, \{2\}, \{3\}\}$ (finest partition)
- $P_1 = \{\{1\}, \{2, 3\}\}$
- $P_2 = \{\{1, 2\}, \{3\}\}$
- $P_3 = \{\{1, 3\}, \{2\}\}$
- $P_4 = \{\{1, 2, 3\}\}$ (coarsest partition)

The Hasse diagram is:



(c) Given:

$$A_1 = \{\{1, 2\}, \{3\}, \{4, 5\}\}$$

$$A_2 = \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$$

- **Greatest Lower Bound** : The meet is the finest partition that is coarser than both A_1 and A_2. We find it by taking pairwise intersections:
  - The element 1 is merged with 2 in A_1, but forms a separate block in A_2. Therefore, 1 must form a separate block on its own.
  - The element 2 is merged with 1 in A_1 and with 3 in A_2. Therefore, 2 must form a separate block on its own.
  - Through similar analysis, it can be concluded that all elements form separate blocks.

$$\text{GLB} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$$

- **Least Upper Bound (LUB)**: The join is the coarsest partition that refines both A_1 and A_2. We find it by taking connected components of the union:
  - The block $\{1, 2\}$ of A_1 and the block $\{2, 3\}$ of A_2 intersect due to the element 2, and they are merged into $\{1, 2, 3\}$.
  - The block $\{4, 5\}$ of A_1 is split into $\{4\}$ and $5\}$ in A_2. However, since the LUB needs to be coarser than A_1, the block $\{4, 5\}$ is retained.

$$\text{LUB} = \{\{1, 2, 3\}, \{4, 5\}\}$$